THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, October 27, 1900, extending through the usual morning and afternoon sessions. Thirty-two persons were in attendance, including the following twenty-five members of the Society:

Professor E. W. Brown, Professor F. N. Cole, Dr. J. W. Davis, Dr. W. S. Dennett, Professor T. S. Fiske, Mr. A. S. Gale, Miss Ida Griffiths, Miss Carrie Hammerslough, Dr. A. A. Himowich, Professor Harold Jacoby, Mr. S. A. Joffe, Dr. Edward Kasner, Mr. C. J. Keyser, Dr. G. H. Ling, Dr. Emory McClintock, Dr. G. A. Miller, Professor R. W. Prentiss, Dr. P. L. Saurel, Professor P. F. Smith, Professor J. H. Van Amringe, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor L. A. Wait, Miss E. C. Williams, Professor R. S. Woodward.

Professor Thomas S. Fiske, Vice-President of the Society, occupied the chair. The Council announced the election of the following persons to membership in the Society: Professor G. L. Brown, South Dakota Agricultural College, Brookings, So. Dak.; Mr. C. H. Davis, New York, N. Y.; Dr. D. N. Lehmer, University of California, Berkeley, Cal.; Miss Ida M. Schottenfels, Chicago, Ill.; Professor F. D. Sherman, Columbia University, New York, N. Y.; Mr. Burke Smith, Northwestern University, Evanston, Ill. Twenty-five applications for membership were received.

It was decided to hold the next summer meeting and colloquium of the Society at Cornell University. A committee, consisting of Professors W. F. Osgood, James Pierpont, J. H. Tanner, and H. S. White, was appointed to coöperate with the Secretary in making the necessary arrangements. The conditions are favorable for extending the colloquium through a period of two or more weeks.

The following papers were read at this meeting:

(1) Professor MAXIME BÔCHER: "On linear dependence of functions of one variable."

(2) Professor DAVID HILBERT: "Ueber Flächen von constanter Gauss'cher Krümmung."

(3) Professor E. O. LOVETT: "Three notes on the geometry of contact transformations."

(4) Dr. G. A. MILLER: "On a theorem in substitutions."

(5) Mr. H. W. KUHN: "Several theorems on imprimitive groups."

(6) Professor S. L. PENFIELD; "The solution of spherical triangles by graphical methods, with exhibition of scales and protractors."

(7) Miss I. M. SCHOTTENFELS: "On a set of definitional functional properties for the analytical function

$$f(z) = \frac{\tan \pi z}{\pi}."$$

(8) Professor P. F. SMITH: "Geometry within a linear spherical complex."

(9) Dr. E. J. WILCZYNSKI: "Invariants of systems of linear differential equations."

Professor Hilbert's paper was presented to the Society through Professor E. H. Moore, Mr. Kuhn's through Dr. G. A. Miller, and Miss Schottenfels's through the Secretary. Professor Penfield was introduced by Professor P. F. Smith. In the absence of the authors, Mr. Kuhn's paper was read by Dr. Miller, and the papers of Professor Bôcher, Professor Hilbert, Professor Lovett, Miss Schottenfels, and Dr. Wilczynski were read by title.

Professor Bôcher's paper appears in the present number of the BULLETIN. Abstracts of the remaining papers are given below.

Professor Hilbert's paper will appear in an early number of the *Transactions*. Following Beltrami, it is known that a piece free of singularities of a surface of negative constant curvature makes real in the domain of ordinary euclidean space a piece of a Lobachevsky (non-euclidean) plane, the straight lines of the Lobachevsky plane corresponding to the geodetic lines of the surface of negative constant curvature and the distances and angles in the plane being the real distances and angles on the surface. Of the known surfaces of negative constant curvature there is none which in all its extent is continuous and has a continuously varying tangent plane; on the contrary all such known surfaces have singular lines beyond which a continuation of the surface with preservation of the properties just mentioned is impossible. The first part of this memoir concludes with the theorem that there exists no surface of negative constant curvature which in all its extent is continuous and has a continuously varying tangent plane, and hence that the complete Lobachevsky plane cannot by the method of Beltrami be realized in ordinary euclidean space.

The principal theorem of the second part is this: On any simlpy or multiply connected closed region entirely in the finite and free of singularities lying on an analytic surface of positive constant curvature + 1 the greater of the two principal radii of curvature of the surface at the various points of the region takes its maximum value at a point of a boundary, and at an interior point only if the surface is a part of the sphere of radius 1. From this follows immediately a new proof of the theorem proved by Liebmann (*Göttinger Nachrichten*, 1899) that the sphere of radius 1 is the only closed analytic surface free of singularities and of positive constant curvature +1.

Professor Lovett's paper consists of three notes. The first is a collection of original problems relative to the contact transformations between the principle elements of ordinary space, namely points, planes, straight lines, and spheres, and has been sent to the *Annals of Mathematics* for publication.

The second constructs the invariants of an n dimensional space as was done for four dimensional space in a recent number of the BULLETIN. It has been offered to the *American Journal of Mathematics*.

The third determines all contact transformations between singular manifoldnesses of surface elements and employs the results to effect the integration of the equations defining certain other categories of transformations. A surface element is the ensemble of a point and a plane through it, the dimensions of the plane being infinitesimal. Two surface elements are said to be associated when the point of each lies in the plane of the other to terms of the first order. An element manifoldness of surface elements is an aggregate such that every element of the aggregate is associated with its infinitely near neighboring element. These notions of surface element and element manifoldness were introduced by Lie. Now among these element manifoldnesses there exist certain ones which enjoy the property that every element of the aggregate is associated not only with its nearest neighbor but also with every other element of the family. Such element manifoldnesses might be called singular. There are but three varieties of such manifoldnesses, namely, the aggregates of all the surface elements of a point, of all those appertaining to a straight line, and of all the elements belonging to a plane. If it be proposed then to determine all the contact transformations which change singular element manifoldnesses into such,

the following cases call for further examination: 1° those which change point into point; 2° those which change point into point and plane into plane; 3° those which change point into plane and plane into point; 4° those which change straight line into straight line; 5° those which change plane into plane. The first three categories are well known to consist respectively of the infinite group of all extended point transformations, of the finite group of all projective transformations, and of all dualistic transformations. The fourth category is equally well known to consist of the second and third categories, a result which yields itself immediately from the above definition of singular element manifoldness. The fifth category which does not seem to have been constructed is readily designed geometrically from the products DPD, where D is the general dualistic transformation and P is an arbitrary point transformation.

Dr. Miller's paper is in abstract as follows: Let s_1, s_2 be any two substitutions which involve the same n letters, and let G_1, G_2 be the groups which are composed of all the substitutions in these n letters that are commutative with s_1, s_2 respectively. The necessary and sufficient condition that G_1, G_2 represent the same substitution group is that s_1, s_2 are similar. All the substitutions of degree m that are commutative with s_1 form a group. When $m = n + b$ and this group involves all the letters of s_1, it is the direct product of G_1 and the symmetric group of degree b. Two dissimilar substitutions of degree n give rise to two distinct groups of degree m. Hence every substitution s_1 gives rise to an infinite number of substitution groups, each of which is distinct from the infinite number of groups that are obtained by the same process from any substitution that is not similar to s_1.

Mr. Kuhn's paper relates to imprimitive groups that contain more than one set of systems of imprimitivity. Let G denote any regular group. It is proved that the number of sets of systems of imprimitivity that can be found for G is equal to the number of its subgroups (not including identity). The cyclical group of order p^2 is therefore the only regular group that contains just one set of systems of imprimitivity. Any set of systems that corresponds to a subgroup that contains no invariant subgroup of G besides identity is permuted by the substitutions of G according to a transitive group that is simply isomorphic to G; any

other set is permuted according to a transitive group that has a $(1, \alpha)$ isomorphism to G, α being greater than unity.

Let G' denote any imprimitive group that has more than one set of systems of imprimitivity. When the substitutions of G' permute all its sets of systems according to primitive groups and when none of the heads is identity, then

(1) No two of the heads of G' can contain a common substitution besides identity, and hence

(2) Each substitution of any other given head is commutative to each substitution of any other head.

(3) Each head contains at least one substitution whose degree equals the degree of G', and

(4) Each head is formed by establishing a simple isomorphism among its transitive constituents.

After a brief statement of the principles of the stereographic projection Professor Penfield exhibited a number of devices for plotting. One of these consisted of sheets upon which were printed a graduated circle of 14 cm. diameter and four scales, as follows: No. 1, giving the radii of stereographically projected arcs of great circles. No. 2, giving the radii of stereographically projected arcs of vertical small circles. No. 3, giving the degrees of a vertical circle stereographically projected upon a diameter. No. 4, giving the radius of the circle divided into one hundred parts. Other devices consisted of two protractors, one having along its base line a scale giving the degrees of a circle stereographically projected, the other used in measuring and for that purpose printed on transparent celluloid, giving the arcs of vertical small circles stereographically projected. There were also exhibited devices of a similar nature, but made up on a large scale for blackboard demonstrations.

Right and oblique angled triangles were solved graphically on the blackboard, and the graphical solution of similar problems carefully executed on the printed sheets were exhibited, indicating that within a circle of 14 cm. diameter computations could be made with an average error not exceeding seven minutes.

There were also exhibited a patented arm protractor for drawing and measuring plane angles, and goniometers especially devised for measuring crystals.

Special emphasis was laid upon the importance of graphical methods for making the studies of plane and spherical trigonometry and geometry more real to students, as a

means of checking numerical calculations, and as a rapid method for obtaining approximate results.

The subject of Miss Schottenfels's paper was suggested by the investigation of Professor E. H. Moore for the function $\frac{\sin \pi z}{\pi}$ in the *Annals of Mathematics*, volume 9, January, 1895. A set of definitional functional properties are obtained for the function $f(z) = \frac{\tan \pi z}{\pi}$, and it is shown that there is one and only one function $f(z)$ possessing these properties. The external exponential factor is then determined in the expression of the function as a Weierstrassian infinite product of primary factors.

Professor Smith's paper takes up the relation of the geometry of reciprocal radii to the geometry within a spherical complex, as determined by the single isomorphism of the corresponding groups, viz., these groups are transformed into each other by contact transformation, whose properties were defined and studied by the author in the paper published in the *Transactions*, October, 1900, "On surfaces enveloped by spheres belonging to a linear spherical complex."

In a paper, soon to be published in the *American Journal of Mathematics*, Dr. Wilczynski has shown that the most general point transformation which converts a general system of n linear homogeneous differential equations into another of the same form and order is

$$(1) \qquad x = f(\xi), \quad y_k = \sum_{i=1}^{n} a_{ki}(\xi)\eta_i, \qquad (k = 1, 2, \cdots, n),$$

where $f(\xi)$ and $a_{ki}(\xi)$ are arbitrary functions of ξ.

Functions of the coefficients and of their derivatives which have the same value for the original as for the transformed system are called invariants. Covariants are such invariant functions involving also $y_1, \cdots, y_n, y_1', \cdots, y_n'$, etc.

Equations (1) define an infinite group G in the $n+1$ variables $x, y_1, \cdots, y_n$. A subgroup of this is that for which x remains unchanged and only $y_1, \cdots, y_n$ are transformed. Functions of the coefficients invariant under this subgroup are called *seminvariants*. These can be found by setting up, by the method of infinitesimal transformations, the partial differential equations which they satisfy. The invariants are functions of the seminvariants.

The actual computation of the seminvariants and invariants is carried out only for a system of two equations each of the second order, *i. e.*, for the system

$$y_i'' + p_{i1}y_1' + p_{i2}y_2' + q_{i1}y_1 + q_{i2}y_2 = 0, \qquad (i = 1, 2), \tag{2}$$

but the methods employed admit of generalization.

If we put

$$\begin{aligned} u_{11} &= 2p_{11}' - 4q_{11} + p_{11}^2 + p_{12}p_{21}, \\ u_{12} &= 2p_{12}' - 4q_{12} + p_{12}(p_{11} + p_{22}), \\ u_{21} &= 2p_{21}' - 4q_{21} + p_{21}(p_{11} + p_{22}), \\ u_{22} &= 2p_{22}' - 4q_{22} + p_{22}^2 + p_{12}p_{21}, \end{aligned} \tag{3}$$

then two seminvariants are

$$I = u_{11} + u_{22}, \quad J = u_{11}u_{22} - u_{12}u_{21}. \tag{4}$$

Four quantities v_{ik} are now formed, cogredient with the quantities u_{ik}, viz.,

$$\begin{aligned} v_{11} &= 2u_{11}' + p_{12}u_{21} - p_{21}u_{12}, \\ v_{12} &= 2u_{12}' + (p_{11} - p_{22})u_{12} - p_{12}(u_{11} - u_{22}), \\ v_{21} &= 2u_{21}' - (p_{11} - p_{22})u_{21} + p_{21}(u_{11} - u_{22}), \\ v_{22} &= 2u_{22}' - p_{12}u_{21} + p_{21}u_{12}, \end{aligned} \tag{5}$$

and a new independent seminvariant is

$$K = v_{11}v_{22} - v_{12}v_{21}. \tag{6}$$

A third set of quantities w_{ik} is formed from v_{ik} and p_{ik} in exactly the same way as v_{ik} from u_{ik} and p_{ik} in (5), i. e.,

$$\begin{aligned} w_{11} &= 2v_{11}' + p_{12}v_{21} - p_{21}v_{12}, \\ &\text{etc., etc.;} \end{aligned} \tag{7}$$

and since the v_{ik}'s were cogredient with the u_{ik}'s, so are the w_{ik}'s. Then

$$L = w_{11}w_{22} - w_{12}w_{21} \tag{8}$$

is a new seminvariant.

All seminvariants are functions of I, J, K, L, and their derivatives.

Some of the invariants are

$$\theta_4 = I^2 - 4J \quad ,$$

$$\theta_6 = 2I(I^2 - 4J) + 5(K - I^2) + 4(K - 2J'' + II').$$

etc., these being of weight 4 and 6 respectively.

If the system of equations (2) is equivalent to a single linear differential equation of the second order, all of the invariants vanish, and conversely.

Any system of form (2) can be transformed into another for which $p_{12} = 0$. This is called the semicanonical form of the system. The subgroup G' of G which leaves this form unchanged is examined. But we can fulfill the further condition $q_{11} + q_{22} = 0$, and a system for which both $p_{12} = 0$ and $q_{11} + q_{22} = 0$, is said to be in the canonical form. The subgroup G'' of G' which leaves this unchanged is a finite group of very simple form, and has some additional invariants of the form called quadri-derivatives by Forsyth.

Only a few simple results about covariants are mentioned. This and further generalizations are left for a future paper.

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ON LINEAR DEPENDENCE OF FUNCTIONS OF ONE VARIABLE.

BY PROFESSOR MAXIME BÔCHER.

It is ordinarily stated that the identical vanishing of the determinant

$$D = \begin{vmatrix} u_1 & u_2 & \cdots u_n \\ u_1' & u_2' & \cdots u_n' \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ u_1^{[n-1]} & u_2^{[n-1]} & \cdots u_n^{[n-1]} \end{vmatrix}$$

is a sufficient* condition for the linear dependence of the functions $u_1(x), u_2(x), \cdots, u_n(x)$. This is perfectly true if the u's are analytic functions of the complex variable x. *This condition is however no longer sufficient if we are dealing with functions of a real variable,* even though these functions possess derivatives of all orders for every real value of x. The truth of this statement will be seen from the example of two functions u_1 and u_2 defined as follows:

$$u_1 = \begin{cases} e^{-1/x^2} & (x + 0) \\ 0 & (x = 0) \end{cases} \qquad u_2 = \begin{cases} e^{-1/x^2} & (x > 0) \\ 0 & (x = 0) \\ 2e^{-1/x^2} & (x < 0) \end{cases}$$

* It is of course a necessary condition provided the u's have derivatives of the first $n - 1$ orders.

Similar examples for a larger number of functions can readily be built.

The following theorem however is true:

If $u_1, u_2, \cdots, u_n$ are single valued functions of the real variable x defined at every point of a certain interval and having at every point of this interval derivatives of the first $n-1$ orders, and if it is possible to strike out the last row and one of the columns of the determinant D in such a way that there is no point of the interval in question at which the remaining determinant and its derivative both vanish, then if D vanishes at every point of the interval, the functions $u_1, u_2, \cdots, u_n$ will be linearly dependent throughout this interval.

This theorem can be readily proved by a slight extension of the method given for instance by Heffter in his book on linear differential equations p. 233.

EMS, GERMANY,
September 15, 1900.

[*Note added November 2, 1900:* I have just found in Pascal's book on determinants a reference to three papers by Peano (*Mathesis*, vol. 9 (1889), p. 75 and p. 110; *Rend. d. Accad. d. Lincei*, ser. 5, vol. 6 (1897), 1° sem., p. 413), in which the question which I have here considered is taken up. My result is however different from Peano's, which states that the identical vanishing of D is a sufficient condition for linear dependence, provided there is no point at which the first minors corresponding to the elements of the last column all vanish.]

REPORT ON THE GROUPS OF AN INFINITE ORDER.

BY DR. G. A. MILLER.

(Read before Section A of the American Association for the Advancement of Science, New York, June 28, 1900.)

VARIOUS terms have been employed to designate the smallest elements of which any abstract group is composed. Cayley has called them symbols,* or symbols of operation. Dyck and many others have called them operations† or operators. Frobenius and others have called them elements.‡ In what follows we shall employ the last one of

*Cayley, *Phil. Magazine*, vol. 7 (1854), p. 41.
†Dyck, *Math. Annalen*, vol. 20 (1882), p. 1.
‡Frobenius, *Crelle*, vol. 86 (1879), p. 218.

these terms. According to the early definitions of an abstract group a set of distinct elements becomes a group with respect to a given law of combination when it has the property that no new element is obtained by combining the elements of the set in every possible manner according to this law of combination.* In recent years the following additional conditions have been imposed:

(1) The elements must be associative when combined according to the given law. (2) From either of the two equations $s_1s_2 = s_1's_2$, $s_2s_1 = s_2s_1'$ it follows that $s_1 = s_1'$. (3) If any two of the elements of the equation $s_1s_2 = s_3$ are in the given set there is one and only one element in the set which may be used for the third. This definition is much more explicit than Cayley's dictum† "a group is defined by the law of combination of its symbols," since it is easy to give definite laws of combination of a set of distinct elements which do not form a group. Every particular group is, however, defined by the special laws of combination of its symbols—by its own multiplication table.

One of the best known examples of a group of an indefinitely large order is furnished by the totality of finite integers when they are combined by addition. If we denote by s^{-a} the inverse of s^a, it is clear that the negative exponents of elements may be used to indicate subtraction and that the identical element is $s^0 = 0$, when the elements of a group are combined by addition. In general, if s represents any integer, the exponent of s will indicate the number of times this number is to be added or subtracted. The given group of indefinitely large order is evidently generated by $s_1 = 1$ as well as by $s_2 = s_1^{-1} = -1$, but by no other one of its elements. While $\varphi(m)$ of the elements of any cyclical group of a given finite order m are generators of the group we have here a cyclical group of an indefinitely large order which contains only two generating elements. All the multiples of any integer whose absolute value exceeds unity constitute a subgroup of this group and each one of these subgroups is of an indefinitely large order and involves only two elements that generate it.

In order to obtain a subgroup of a finite order m we may take the complete series of smallest positive residues mod. m $(0, 1, 2, 3, \cdots, m-1)$. When we add any two numbers of this series and use the smallest positive residue in place of the sum we evidently obtain some number of this series.

*Cayley, loc. cit.; Lie, Le centenaire de l'École normale, 1895, p. 486; Klein, Vorlesungen über das Ikosaeder, 1884, p. 5.

†Cayley, *Amer. Jour. of Math.*, vol. 1 (1878), p. 51.

As the other group conditions are satisfied we have here a cyclical group of order m which is generated by any one of the $\varphi(m)$ numbers that are not greater than m and prime to it. Hence we observe that the finite integers, when combined by addition, furnish very elementary examples of groups of any given finite orders as well as of groups of indefinitely large orders.

Since x cannot be an integer in such an equation as $\delta x = 19$, it follows that the integers do not constitute a group with respect to multiplication.* The finite commensurable numbers which exceed zero, on the contrary, constitute such a group. Those which are less than zero do not constitute a group; but, together with those that exceed zero, they clearly constitute a group with respect to the given operation. This group of infinite order is included in the one formed by all the real numbers with the exception of 0 and ∞. The latter group is, in turn, included in the abelian group formed by multiplying together all the complex numbers whose absolute value is neither 0 nor ∞. The finite commensurable numbers, including 0, evidently constitute a group of infinite order when they are combined by addition and this group is included in the one formed by all the real finite numbers. The totality of complex numbers of finite absolute value constitute a group, with respect to addition, of which the preceding is a subgroup.

Although all the integers do not form a group with respect to multiplication, yet it is clear that all the powers of any finite integer, with the exception of 0 and 1, constitute a group of an indefinitely large order with respect to this operation. In order to obtain a group of a finite order by multiplying integers together we may take the $\varphi(m)$ positive numbers which are less than m and prime to it and replace their products by the least positive residues mod. m. We thus obtain a very important class of abelian groups which has been studied recently by Weber and others.†

While the fundamental operations addition, multiplication, and their inverses furnish such lucid examples of groups of an infinite as well as of a finite order, yet it is not customary to claim that the theory of groups is as old as these elementary operations. At least one group property of these operators must have been observed very early; viz., that the successive application of two elements is equivalent to the single application of some one of them. While

* Weber, Lehrbuch der Algebra, 1899, p. 4; cf. ibid., 1896, p. 54.

† Weber, Lehrbuch der Algebra, 1899, p. 60. Cf. Burnside, Theory of groups of a finite order, 1897, p. 239.

this concept is of great theoretic interest yet it did not enter explicitly into the early use of these fundamental operations and hence it may perhaps be said to have played a secondary rôle in the developments along this line.

The concept of group of an infinite order appears in many other early developments but the earliest extensive paper in which this concept occupies a very prominent place seems to be the "Mémoire sur les groupes de mouvements," by Jordan, which was published in the *Annali di Matematica*, vol. 2 (1868).* No general definition of a group occurs in this memoir but on page 181 a "définition caractéristique d'un groupe de mouvements" is given, which is in accord with the present definition. An abstract of this memoir was published somewhat earlier in the Paris *Comptes rendus*, vol. 65 (1867). In the beginning of the memoir Jordan points out that the determination of all the groups of movements is equivalent to the determination of all the possible systems of molecules which can be superposed in different positions.

Bravais had studied particular cases of this question at a much earlier date from the latter point of view,† and applied his results in the study of the structure of crystals. Since Jordan's memoir was inspired by the work of Bravais it might perhaps be said that the theory of groups of an infinite order has its source in the theory of the structure of crystals. However, the group concept does not occupy as prominent a place in the works of Bravais as in the mentioned memoir of Jordan. We proceed to give an elementary exposition of the first part of this memoir with a view to giving an accurate idea of its contents. After some preliminary remarks and a brief study of the composition of movements the author proceeds to determine all the groups of translations in the following manner:

The lengths and the directions of all the translations of the group are represented by line segments starting from a general point P. It is first assumed that all of these segments are of finite lengths and the length of the shortest segment L_1 (or one of the shortest segments, if there is more than one of minimum length) is represented by l. The line through L_1 is taken for the x-axis. An indefinite number of segments of length l are laid off on the x-axis starting from P. The point P can evidently be transformed into

* Cf. Fricke und Klein, Automorphe Functionen, vol. I, 1897, p. 12. Klein seems to have been the first to make prominent use of the concept of discontinuous groups of an infinite order in function theory; cf. Dyck, *Math. Annalen*, vol. 20 (1882), p. 2, footnote.

† Bravais, *Liouville's Journal*, vol. 14 (1849), p. 167.

any one of these points of division on the x-axis by means of some translation of the group. If P could be translated into any other point on the x-axis, its distance from P would be $ml \pm r$, m being an integer and $0 < r \leqq l/2$. If we combine this translation with the translation through $-ml$, which is also in the group, we obtain a translation whose length cannot exceed $l/2$. As this is contrary to the hypothesis, it follows that the points of division can be transformed into none of the points on the x-axis except points of division, by all the translations of the group.

After proving that all the translations parallel to the x-axis are generated by a single translation the author proceeds to determine all the possible translations in the plane and proves that all of these are generated by two translations which make a finite angle with each other. This is done as follows: Among the given line segments starting from P we take one L_1 whose extremity is at a minimum distance from the x-axis and represent this distance by δ. The first object is to prove that δ must be finite. We may assume that the projection a of L_1 upon the x-axis does not exceed $\frac{1}{2}l$ since the group involves translations of length l parallel to the x-axis. Since the length of L_1 cannot be less than l, we have

$$l^2 \geqq \sqrt{a^2 + \delta^2} \geqq \sqrt{\frac{l^2}{4} + \delta^2}, \text{ or } \delta^2 \leqq \frac{3}{4} l^2.$$

On the line through L_1 an indefinite number of segments of the same length as L_1 are laid off, starting from P. Through the points of division lines are drawn parallel to the x-axis and through the given points of division of the x-axis lines parallel to L_1 are drawn. The plane is thus divided into parallelograms, as is usually done in the study of doubly periodic functions. It is then proved that the vertices of these parallelograms can be transformed only among themselves by all the translations of the group and hence there is no finite translation in the plane determined by L_1 and L_2 except those which are generated by the translations represented by the segments L_1 and L_2. From this it follows that *there are two and only two groups of translations, involving only finite translations, in a given plane.* In one of these groups all the translations are parallel to a given line; in the other they are generated by two translations making a finite angle with each other.

If a group contains any finite translations in addition to those given above they cannot lie in the plane determined

by L_1 and L_2. We choose one of them, having a minimum projection β upon the normal to the given plane, and represent it by L_3. The length of β may be proved to be finite by the following method. The projection of L_3 upon the given plane may be assumed to lie in any one of the four parallelograms which have P for a common vertex. This parallelogram is so chosen that the distance from the x-axis to the terminus of the given projection is $\leqq \delta/2$. Since L_3 cannot terminate closer to the x-axis than L_2 it follows that

$$\delta \leqq \sqrt{\beta^2 + \frac{\delta^2}{4}}, \quad \text{or} \quad \beta^2 \geqq \frac{3}{4}\delta^2.$$

Space is now divided into parallelepipeds whose vertices are the points into which P may be translated by all the operators of the group generated by the translations represented by L_1, L_2, L_3. It is readily proved that a vertex of such a parallelepiped cannot be transformed into any point except a vertex by means of the translations of the group. This is therefore the largest group composed entirely of finite translations in ordinary space and the two groups mentioned above are subgroups of this one. Every point of space which is not a vertex of the given system of parallelepipeds will clearly be transformed into a vertex of a similar system of parallelepipeds by the translations of this group.

There are six groups of translations that involve indefinitely small translations. The first of these is generated by a single infinitesimal translation, the second by an infinitesimal and a finite translation not in the same direction, the third by an infinitesimal and two finite translations the three translations being not co-planar, the fourth is generated by two infinitesimal translations not in the same direction, the fifth by two infinitesimal and one finite translation not in the same plane, and the sixth by three infinitesimal translations not in the same plane. The last of these groups includes all the others as subgroups and each one of these nine groups is of an infinite order.

After the groups which involve only translations have been determined the author proceeds to the groups which are composed of rotations only. Here the matter becomes somewhat more difficult. The first step is to prove that all the axes of rotation must go through the same point. This is done as follows: Let R_1, R_2 be two such rotations. Their combination $R_1 R_2 = R_3$ is a new rotation around some axis A_3. Let x be a point on A_3. R_1 will bring x to some new point x_1. Hence the axis of R_1 must lie on the plane per-

pendicular to the segment joining x and x_1 at its middle point. Since R_2 must transform x_1 into x its axis must lie in the same plane. Hence the axes of any two rotations must go through the same point. Since this point is transformed into itself by the two rotations R_1, R_2 it must also lie on A_3. As A_3 passes through the point x, which is not in the plane of the axes of R_1 and R_2, the three axes of rotation cannot lie in the same plane. Since any other axis of rotation has to meet each of these three lines it must go through their common point, as the lines are not in the same plane. This proves that all the axes of a group composed entirely of rotations must be concurrent.

It is not difficult to see that there are at least eight groups which are composed of rotations only. The first is generated by the rotation through an angle $2\pi/n$, n being any integer. The second is generated by rotating through an infinitesimal angle. The third and fourth are generated by adding to each of the preceding a rotation through 180° around an axis perpendicular to the given one. The fifth, sixth and seventh are the well known rotations which transform the regular tetrahedron, the cube, and the regular icosahedron* into themselves. These groups are respectively of order 12, 24, and 60, and may be represented as the alternating group of four letters, the symmetric group of four letters, and the alternating group of five letters respectively. The eighth is composed of all the possible rotations around a point. The proof that there is no other group of rotations is somewhat lengthy, and we shall not enter upon it.

The greater part of the memoir under consideration is devoted to groups which involve both rotations and translations. These have recently been studied by Schönflies, who published his results in two well-known memoirs.† It may be remarked that Jordan's memoir is not entirely free from errors. Sohncke seems to have been the first to observe that an important class of groups was overlooked in the memoirs mentioned.‡ The determination of all the continuous groups of euclidean and non-euclidean movements in a space of three dimensions was first published by Lie.

Although groups of an infinite order were studied before Lie and Klein began to publish on this subject, yet they were the first to bring them prominently before the pub-

* Cf Klein, Vorlesungen über das Ikosaeder, pp. 1–19.

† *Math. Annalen*, vol., 28, pp. 319–342; vol. 29, pp. 50–80; cf. Schönflies, Chicago Mathematical Congress Papers, 1896, p. 341.

‡ Theorie der Krystallstruktur, 1879, p. 26.

lic.* Their investigations along this line are so far reaching and so numerous as to deserve a report by themselves. Happily the German Mathematical Association has arranged for a report on Lie's work, which is expected to be presented at the meeting of next year (1901). The report is to be prepared by Engel assisted by Scheffers, Schur, and Kowalewski —men who are in the best possible position to prepare an extensive and reliable report in reference to the work of this great Norwegian mathematician. In view of this fact we shall not consider the works of Lie and his followers in the present report.

In 1882 two remarkable memoirs. dealing with groups of an infinite order, appeared. One of these was published by Dyck in the *Mathematische Annalen*, volume 20, and the other by Poincaré in the *Acta Mathematica*, volume 1. These articles have received so much attention that it seems unnecessary to give an outline of them. Burnside has devoted Chapter XII. of his Theory of groups of a finite order (1897) to an exposition of Dyck's memoirs, while many parts of Poincaré's memoir are elucidated in the work on Automorphic Functions by Fricke and Klein, which also appeared in 1897.

This last work makes a clear distinction between continuous groups and groups which contain infinitesimal transformations. It is easy to prove that a continuous group always involves infinitesimal transformations, for if T is any transformation of the group and T' the transformation obtained by using parameters which differ by an infinitesimal from those of T then $T'\,T^{-1}$ is infinitesimal. However the existence of infinitesimal transformations does not prove the group continuous. Attention is called † to the fact that in Lie-Engel's work on Transformationsgruppen and in Poincaré's memoir not sufficient emphasis is laid upon this distinction. The work of Fricke and Klein on the group-

* Cf. Klein, Erlangen Programme (1872), BULLETIN, vol. 2 (1893), p. 215. Lie's contributions are systematically treated in the three volumes of his Theorie der Transformationsgruppen (1888–93); Differentialgleichungen (1891); Continuierliche Gruppen (1893); Berührungstransformationen, vol. 1 (1896); Klein's Einleitung in die höhere Geometrie (1893); etc. A considerable part of Klein's work along this line is contained in the two volumes of his Modulfunctionen (1890–92), and in the Automorphe Functionen, vol. 1 (1897). Numerous publications by Picard and Poincaré have also contributed very much towards the general interest in this subject. Picard was the first to develop the theory of linear differential equations parallel with the Galois theory of algebraic equations. Vessiot, Drach, and others have perfected this theory in certain directions.

† Page 65.

theoretic foundation of automorphic functions is doubtless the best work on discontinuous groups of an infinite order that has yet been published. The direct object of this work is the investigation of the groups of linear substitutions in one variable which do not contain any infinitesimal substitution.

While in Lie's theory the parameters of the product of two transformations are functions of the parameters only of these transformations, Halphen has considered groups in which these parameters are functions both of the variables and of the parameters of the two transformations that are multiplied together. Lie pronounced this group concept trivial and said that it had not found any application.* He expressed the opinion that the definition of group of transformations had perhaps reached a definite form, but that it was not certain that the general notion of group had as yet assumed such a form.†

Study and Engel seem to have discovered independently ‡ that the special linear homogeneous group of the plane contains transformations which are not generated by any infinitesimal transformations of the group, as was apparently assumed by Lie.§ Such transformations have been called *singular transformations* and they have received considerable attention in recent years. The two memoirs by Engel have been followed by numerous other memoirs especially by Taber,‖ who established the existence of singular transformations in the group of orthogonal substitutions in n variables, $n > 3$, and also in the group of linear automophic transformations of a general bilinear form. He has also determined many properties of groups that contain such transformations.

One of his students, Mr. Rettger, has recently investigated ¶ the two and three parameter subgroups of the general projective group in two variables and of the general linear homogeneous group in three variables with regard to singular transformations, and proved that singular trans-

*Theorie der Transformationsgruppen, vol. 3 (1893), p. 19; cf. *Amer. Jour. of Math.*, vol. 11 (1889), p. 182.

† *Le centenaire de l'Ecole normale*, 1895, p. 486.

‡ Engel, *Leipziger Berichte*, vol. 44 (1892), pp. 277–296 and vol. 45 (1893), pp. 659–696; cf. BULLETIN, vol. 3 (1893), p. 66.

§ Cf. Taber, BULLETIN, vol. 6 (1900), p. 199.

‖ Taber, *Amer. Jour. of Math.*, vol. 16, p. 130; *Proc. of the Lond. Mat. Soc.*, vol. 26 (1895), p. 364; *Math. Annalen*, vol. 46, p. 561; BULLETIN, vol. 6 (1900), p. 199. Numerous other references are contained in these articles.

¶ Rettger, *Amer. Jour. of Math.*, vol. 22 (1900), p. 60.

formations occur among the transformations of many of these groups. A list of such subgroups is given at the end of the article. Many of the results of this article are contained in an earlier note by the same author in the *Proceedings of the American Academy*, vol. 33 (1898), p. 493.

Besides the groups which are explicitly of an infinite order there is a large class of groups whose orders depend upon one or more parameters, which are generally assumed to be finite but may also be regarded as infinitely large. To this class belong the symmetric and the alternating groups of degree n,* the cyclical groups of order n, the metacyclic and the semi-metacylic groups, of orders $p(p-1)$ and $\frac{1}{2}p(p-1)$ respectively, etc. Such groups are, however, generally classed with the groups of a finite order. We observed above that Burnside devoted a small part of his theory of groups of finite order to the study of groups of an infinite order. In the second volume of Weber's Algebra these groups receive much more attention. Considerable portions of the chapters on linear groups are devoted to considerations which explicitly relate to groups of an infinite order. The object seems to be to devote some space to discussions in which the order is not restricted and then to proceed to the special cases where the order is finite.

The theory of groups of an infinite order is closely related to that of a finite order and the development of the former has been greatly influenced by the latter. The latter, in turn, has contributed very much towards the interest and importance of the former, and, in several instances, it has led to important new developments in this theory; *e. g.*, Maillet has shown † how the concept of parameter groups, as used by Lie, can be employed in the theory of substitution groups and has made use of this concept to obtain several general theorems on simply isomorphic groups. In a recent article,‡ Loewy has established the fundamental theorem that a linear substitution group of infinite order which contains at least one substitution whose characteristic equation has no equal roots, must always contain a substitution whose order exceeds any given number. The presence of such a substitution is therefore a necessary and sufficient condition that the order of a linear group is infinite.

CORNELL UNIVERSITY,
June, 1900.

* Dyck, *Math. Annalen*, vol. 22 (1883), p. 72.
† Maillet, *Ann. di Matematica*, vol 23 (1895), p. 199.
‡ Loewy, *Math. Annalen*, vol. 53 (1900), p. 225.

THE STRENGTH OF MATERIALS.

The Strength of Materials. By J. A. Ewing, M.A., B.Sc., F.R.S., M.Inst.C.E. Professor of Mechanism and Applied Mechanics in the University of Cambridge, Fellow of King's College, Cambridge. Cambridge, The University Press, 1899. Pp. vii–xii + 1–246, with 150 illustrations in the text.

This work is intended to supply to students "in modern schools of Engineering" a "knowledge of the Strength of Materials and of its application in design * * * which to be effective must be supplemented by laboratory and drawing-office work." It is desirable that the mathematical elastician should learn what light modern experimental research throws on the validity of the hypotheses at the basis of the mathematical theory, and also that he should know what parts of his subject possess most interest for his more practical contemporaries. On the other hand, it is conceivable that the practical man may derive some advantage from realizing how the mathematical treatment which passes current in everyday life strikes the mathematician. This review thus naturally divides itself into two principal parts, the first dealing with the more experimental portions of the book, the second with the mathematical methods. A preliminary description of the contents of the book will facilitate comprehension.

After a brief preface and a table of contents, pp. vii–xii, chaps. I and II, pp. 1–23, define stress and strain, explain their simpler common types, and treat of the ordinary moduli, or elastic constants, for *isotropic* material. Chap. III, pp. 24–58, treats of ultimate strength and non-elastic strain, dealing mainly with simple tension and compression, describes the phenomena presented during the loading of steel and iron up to rupture, and discusses experiments on the effects of rest or of heating after over-straining. It also gives an account of Wohler's experiments on the "weakening" of material under very frequent repetitions of loading and unloading. Chap. IV, pp. 59–95, describes—with excellent illustrations—a number of testing machines and instruments for measuring extensions and compressions, discusses some methods of determining the modulus of rigidity and gives some numerical results.

Chaps. V, VI, and VII, pp. 96–153, deal mainly with the application of the ordinary Bernoulli-Euler mathematical

theory to beams. Analytical methods are generally followed, but the ordinary graphic methods are also illustrated. Chap. VIII, pp. 154–170, treating of "frames," may be regarded as part of a text book on ordinary statics. Chap. IX, pp. 171–186, treats of struts and columns, first on Euler's mathematical theory, and then from the more empirical standpoint of Gordon or Rankine's formula. Chap. X, pp. 187–203, is denominated "Torsion of Shafts." It is however not confined to this, but touches upon the theory of spiral springs and the "whirling" of rotating shafts. Chap. XI, pp. 204–217, treats of shells and thick cylinders under pressure, with a discussion of rotating rings and thin discs. Chap. XII, pp. 218–237, is devoted to "Hanging Chains and Arched Ribs," including parabolic chains, the common catenary, etc. An appendix, pp. 239–242, gives numerical data as to strengths and densities of ordinary building materials; and an index occupies pp. 243–246.

Experiment.—The fundamental fact underlying the ordinary mathematical theory, and therefore all ordinary girder theory, whether its outward form be graphical or analytical, is Hooke's law that stress is proportional to strain, or—as Professor Pearson puts it—that the stress-strain relation is *linear*. On this point we are told (Art. 15) "A material is elastic * * * if the strain disappears when the stress is removed. Strain which persists after the stress * * * is removed is called permanent set. * * * Actual materials are in general nearly perfectly elastic with regard to small stresses * * * if the applied stress is less than a certain limit, the strain * * * disappears wholly or almost wholly when the stress is removed. * * * The limits of stress within which strain is wholly or almost wholly elastic are called limits of elasticity * * *. (Art. 16) Within the limits of elasticity the strain * * * is proportional to the stress." Later we have the following statement (Art. 32): "Within these limits (of elasticity) we may without serious inaccuracy take the strain as * * * proportional to the stress and as disappearing when the stress is removed. Strictly speaking, absolute proportionality of strain to stress is never found, and probably there is no stress, however small, that does not produce some permanent effect. There is always some slight *hysteresis* or lagging in the relation of strain to stress. * * * But in general this imperfection of elasticity is so slight that * * * up to a certain limit, which is in general pretty well defined, Hooke's Law may be taken as substantially accurate."

A student when reading Art. 15 would be apt to conclude that the only two things known to exist are *elastic strain* disappearing simultaneously with the removal of the stress, and *set* persisting permanently. In Act. 32 he seems, however, expected to distinguish between set and hysteresis (a term which seems equivalent to "Elastische Nachwirkung" or what Pearson calls "Elastic after-strain"). It is to be regretted that the author does not bring out more clearly the theoretical distinction between set which persists and strain which gradually disappears, especially as the practical discrimination between them is so difficult.

Another question is whether the author is well advised in assuming—as he seems tacitly to do—that failure of Hooke's law necessarily implies the existence of hysteresis, or set, and conversely. It is at least conceivable that materials exist in which the relation between stress and *elastic* strain is not linear. Experiments by Hodgkinson, Kupffer, and others have sometimes been supposed to show that cast iron is such a material. Our author refers to Hodgkinson's experiments (Art. 38), but not in such a way as to show whether he is aware of the interpretation put upon them by others (see Todhunter and Pearson's "History of Elasticity" Vol. 1, Note D, p. 891; Vol. 2, Art. 793, etc.). The subject seems one which might repay experimental re-investigation with modern appliances.

The next question that calls for consideration is that which gives its name to the book. In Art. 33 we are told "The load which suffices to cause rupture measures the ultimate strength of the piece." At this stage, however, a difficulty comes in. Engineering calculations are almost all ultimately based on Hooke's law. But when the stress is carried beyond the elastic limit the departure from the law begins to be serious, and with the approach of rupture the law is often a travesty of the facts. If then calculation supplies values for the stresses or strains, at any point of a structure, such as answer to the breaking load, the only legitimate conclusion is that the physical basis for the calculation is not satisfactory. It would certainly be prudent in such a case to conclude that the load contemplated is excessive, or badly distributed; but how excessive it may be, or what would happen if it were applied, lies outside the range of the ordinary theory. From the theoretical standpoint the most logical course would seem to be to consider not the breaking load but that answering to the elastic limit. According to our author, the elastic limit is usually pretty clearly defined in most building materials; and, supposing

this to be true, there would not appear to be serious objection on the practical side to focussing attention on it rather than on the breaking load. It appears, however, to be the invariable or almost invariable custom amongst practical men to make use ostensibly of the breaking load modified by so-called "factors of safety." After giving the definition

factor of safety = (ultimate strength) ÷ (extreme working stress),

our author adds (Art. 34): "The rational use of a factor of safety * * * results * * * in preventing waste of material locally by making the margin of strength equal for all parts." Perhaps a more logical way of presenting the facts would be to say that what the engineer puts before him is the "extreme working stress" which he gets by dividing his ultimate strength, *i. e.*, breaking load, by a certain number which he calls a factor of safety. Supposing the working stress thus arrived at not to strain the material beyond the elastic limit, there is no objection on mathematical grounds to calculations in which it plays a part.

There still, however, remains a difficulty; for how are we to determine the factor of safety? On this point we are told in Art. 34: "The choice of a factor * * * depends on many considerations, such as the probable accuracy of the estimated loads and also that of the theory * * * the possible effects of bad workmanship, * * * the variability or uniformity of the load. * * * The factor * * * also serves to provide for * * * incidental shocks. * * * The factor * * * is rarely less than 3, it is very commonly 4 or 5, and it is sometimes as much (in machines) as 10 or 12." The factor of safety covers, in short, uncertainty in the material, unforeseen contingencies from loading or weather, and gaps in the engineer's physical or mathematical knowledge. So long as all these sources of uncertainty are lumped together, "a rational use of a factor of safety" seems rather much to hope for. One of the things most likely to narrow the range of uncertainties is a knowledge of what really determines the changes that take place in the material as the load is raised. There are various theories bearing on this point. The three principal believe that in isotropic materials the crux is: (1) the maximum principal stress, (2) the maximum stress-difference (or difference between the greatest and least principal stresses), (3) the greatest principal strain. Of these theories the first—employed by Lamé and others—is now hardly supported by any the-

oretical elastician; the second—suggested by Tresca's experiments on the flow of metals—has been used by Prof. G. H. Darwin and others; the third was that advocated by St. Venant. Our author refers explicitly to no theory. In dealing with ultimate strengths he invariably speaks of the *stress*, and his language in many places (e. g., Art. 8 where he says "the greatest principal stress measures the greatest intensity which the material has to bear") certainly suggests that he supports the first or greatest stress theory of rupture. In general, however, he really avoids committing himself. When a piece of given material, of given size and shape, is loaded in a particular way, so long as Hooke's law holds, doubling the load doubles alike every stress and strain; thus the condition of the material can be defined by specifying the magnitude of any one convenient stress or strain. In such a case the laying down a limiting *stress* does not commit one to any theory of rupture. If, for instance, we take a uniform bar under simple tension S, the greatest stress and the maximum stress-difference are both S, and the greatest strain is S/E, where E is a constant for the material (Young's modulus). Thus we may specify that the working stress shall not exceed $\bar{S}$, and yet believe that the critical thing is that the maximum stress-difference shall not exceed $\bar{S}$, or the greatest strain shall not exceed $\bar{S}/E$. Our author refers to separate limiting stresses for simple tension, simple compression, and shearing; and if the simple states were those that always existed, one might get along fairly comfortably without any theory of rupture. As matters actually stand, however, empiricism is under the disadvantage that absolutely simple tension, compression, or shearing are rather ideal states, and are only approximated to even in the testing machine. In some of the problems treated in this book the stress system is not simple, and, as we shall see later, the conclusions drawn as to the strength are not always reliable unless the maximum stress theory be really true.

Nature of material.—In Arts. 17 to 19, Young's and the rigidity moduli are defined in a way applicable only to isotropic materials, while isotropy is first mentioned, and then only incidentally, in Art. 20. In Art. 26, where the stress-strain relations are formulated, it is indeed mentioned that isotropy is assumed, but nothing is said as to the general form of the relations in non-isotropic materials. In Art. 33 there are fuller references to differences between isotropic and non-isotropic materials, but only from the point of view

of ultimate strength. The same article tells us that wrought iron is not isotropic, and data confirming this statement appear in Table III, p. 241. Yet in Table IV, p. 241, there appear values of Young's and the rigidity moduli for wrought iron, as if it were isotropic, and we even have a Young's modulus given for "timber" without any specification of direction. All the mathematical calculations in the book, which depend in any way on the precise form of the stress-strain relations, tacitly assume isotropy; and no attempt is made to ascertain what modification is necessary to render the results applicable to non-isotropic material. This assumption of isotropy is by no means peculiar to the present work. Rankine and St. Venant indeed in various writings intended for practical men treated non-isotropy in considerable detail, but we know of no recent English practical writer who has followed this example. It is very possibly undesirable that elementary books should go into much detail in this matter, but greater prominence should be given to the limitations involved by the assumption of isotropy, whose occurrence is after all probably the exception rather than the rule.

Mathematical treatment.—The author puts *stresses* in the forefront. Supposing the three principal stress directions to coincide with the rectangular axes, then the strain "e_x is made up partly of the direct strain which (the stress) p_x produces in its own direction, and partly of the lateral strains produced by p and p_z" (p. 21). In this way Young's modulus and Poisson's ratio at once come to the front as elastic constants. The general properties of strain, the relations of strain to displacement, the surface stress equations and the ordinary body stress differential equations are not considered. When the basis provided proves too narrow, as it sometimes does—*e. g.*, in Arts. 143 and 147—, the author makes such supplementary assumptions as he finds necessary. This seems a trifle hard on the conscientious student whose knowledge of elasticity is confined to this book. The proof of the "equality of shearing stress in two directions" in Art. 12 is not rigid; the author makes no use of his hypothesis that the cube is indefinitely small, and in fact does not refer at all to bodily forces. The use of $1/\sigma$ for Poisson's ratio may lead to confusion, as that quantity is represented by σ in Thomson and Tait's Natural Philosophy and in Love's Treatise on Elasticity. Again in Art. 147, $1/\sigma$ is replaced by μ — used to represent the rigidity by St. Venant, Pearson, Love, etc. —the rigidity itself being denoted by C. In discussing the

determination of the rigidity in Article 72, μ is used for the twisting couple for unit angle of twist. The difference between static and kinetic moduli is presumably an unnecessary refinement in engineering, so that the absence of reference to the subject in Arts. 71 and 72 is hardly surprising, but in Art. 73 the discussion of Maxwell's method of determining the rigidity might well have called attention to the fact that it keeps the tension of the wire unaltered and eliminates the moments of inertia of the shifted masses round their own axes of figure.

Before entering on a discussion of beam problems in Chap. V, the author describes "uniformly distributed stress," which means stress the same in intensity and direction at every point of a plane surface, and "uniformly varying stress" in which the stresses are parallel, but vary as the distance from some line (a "neutral axis") in the plane to which they are perpendicular. He adds (Art. 79): "Uniformly varying stress is practically important because it occurs * * * in a bent beam, in a tie-rod when subjected to a non-axial pull, and in a long strut or column." In short the author *assumes* the ordinary Bernoulli-Euler theory for beams whether locally or generally loaded. He considers first the stresses alone, explaining several graphic methods. He then proceeds to determine the "slopes" and "deflections" in beams, on the assumption that "The strain on any imaginary filament taken along the length of the beam is sensibly the same as if that filament were directly compressed or extended by itself" (Art 97). Supposing x measured parallel to the length of the beam, and denoting the slope by i, the deflection by u, and the radius of curvature of the central axis by R, the author (Art. 99) deduces $1/R = di/dx = d^2u/dx^2$, whence $i = \int (1/R)dx$, $u = \int i dx$. Having expressed $1/R$ in terms of the bending couple by the Bernoulli-Euler method, he thus arrives by one step at i and by a second step at u. In introducing the formula $1/R = d^2u/dx^2$ the author specifies that the deflection is assumed very small, but he does not explain that this means that $(du/dx)^2$ is neglected as compared to unity.

The strained form of a bent beam is illustrated by Fig. 84, Art. 102. This would have been clearer if the excess of width at the top had been more exaggerated (cf. Thomson and Tait's Natural Philosophy, Art. 716). Also when the author says: "The lateral strain being $1/\sigma$ of the longitudinal strain, the anticlastic or transverse curvature * * * is $1/\sigma$ of the longitudinal," one cannot but fear that

the student will obtain an erroneous idea both of what is meant by "anticlastic curvature" and of what actually takes place. The conclusions in the end of Art. 102 as to the bending of a "wide flat strip" seem hardly in accordance with Pearson and Filon's results (*Quarterly Journal of Pure and Applied Mathematics*, Vol. 31, p. 66, especially Arts. 52 et seq.). The author's treatment of the ordinary first approximation results for beams has much to commend it, especially for students weak at mathematical analysis. When, however, he tries to improve on the first approximation results, he is less satisfactory. Thus in Art. 95 when deducing the distribution of shearing stress over the section of a beam, and in Art. 105, when applying the results so obtained to the calculation of the "additional deflection due to shearing," he tacitly assumes that the shearing stress is the same at every point of a narrow strip perpendicular to the plane of bending. We are not aware that this has been *proved* in any particular case; it is not in accordance with the solution obtained by St. Venant for beams under terminal shearing force (Todhunter and Pearson's History, Vol. 2, Part I, p. 64), or with that found by Pearson and Filon for a heavy elliptic beam (l. c., pp. 88 and 98). Further the "correction" in Art. 105 is of the same order of magnitude as the *difference* between the vertical displacements at different points of the cross section; it was obtained by equating work done by stresses to work done by gravity on the load, while no account was taken of what part of the central section the load was applied at.

In dealing with such matters, a reviewer cannot but regret the great scarcity of references in the work to mathematical text books or papers. There are two or three references to Lord Kelvin and one to Pearson, in connection with experimental work or definitions, while St. Venant is mentioned generally in connection with torsion; but to the text books of these authors, or to that of Love, or to their mathematical papers, there seems no reference anywhere. Thus when the author reaches an unusual result one is uncertain whether it has or has not any more substantial basis than the proof presented in the book. The average engineering student is perhaps unlikely to utilize references, but their presence would at least not harm him and should facilitate enlargement of ideas amongst the more competent.

The treatment of "frames" in Chap. VIII is condensed, and various assumptions,—e. g., in Arts. 113, 116, 117, and

119—seem to require more explanation. The *index* tells us that the "method of reciprocal figures" in Arts. 114 et seq. is due to Maxwell, but his name does not appear in the text. The subject of "redundant members" in frames is very lightly touched upon, and there is no reference to Macaulay's recent extension of Maxwell's method (*Philosophical Magazine*, Jan., 1898, p. 42).

The treatment of struts in Chap. IX is fairly full, but the explanation of Euler's method in Art. 122 might be improved. On p. 172 the curvature is taken as d^2u/dy^2 without restriction. The solution of course assumes that $(du/dy)^2$ is negligible, and yet the conclusion is drawn: "the same force will serve to keep the strut bent whether the curvature is small or not so small." Presumably in engineering practice anything but small curvature is rare, but considerable elastic bending is easily introduced in thin rods, and a reference to some more general treatment (*e. g.*, Love's Elasticity, Vol. 2, Arts. 227 et seq.) would appear desirable. The conclusion reached by the author (Art. 122) that the length of a segment of a bent strut may be any submultiple, even or odd, of the whole length, is correct, but it does not follow from his mathematics. His final equation justifies his conclusion $(P/EI)^{\frac{1}{2}}(L/2) = n\pi/2$, only when n is an *odd* integer. The fact is that in originally assuming $u = u_1\cos\{(P/EI)^{\frac{1}{2}}y\}$ the author excluded all cases in which the displacement u vanishes at the center of the rod (his origin). The illustration selected in Art. 126 of "struts with lateral load" is rather curious. A bending moment whose true value is $(WL/8)\{1 - (2x/L)^2\}$ is represented as $(WL/8)\cos(\pi x/L)$, thus replacing the true differential equation by one of the type

$$(d^2u/dx^2) + au + c\cos bx = 0.$$

A pure mathematician, it may be feared, would hardly accept as adequate the explanations that "a parabola * * * is not far from coincidence with a curve of sines" and that $1 - (2x/L)^2$ agrees in value with $\cos(\pi x/L)$ both when $x = 0$ and when $x = L/2$. It may be added—as a personal contribution from the reviewer—that while the solution of the *true* differential equation presents no real difficulty, the result does not lend itself to manipulation by quite elementary methods.

Chap. X calls for a few criticisms. Art. 127 assumes—as is indeed usual—that when a circular shaft of any length is twisted in any symmetrical way each element suffers a

displacement directly proportional to its distance from the axis. There are in reality any number of possible symmetrical kinds of twisting, the species depending on the force system applied, and unless the length be great compared to the diameter there is no *a priori* reason to suppose that the stress and strain will even approximate to the ordinary simple law (see *Cambridge Transactions*, vol. 14, p. 365). The crank shaft dealt with in Art. 131 is a case where the application of the Bernoulli-Euler method is somewhat speculative. The author seems here to hesitate between the greatest principal stress and the maximum shearing stress as the criterion for rupture, but his language is rather ambiguous. In Art. 133, dealing with torsion beyond the elastic limit, the author supposes that the twisting couple can be increased until practically the whole of the material—starting with the outer layers—passes beyond the elastic limit, and that the altered material possesses everywhere the same shearing stress q'; he speaks of q' as "the ultimate shearing stress of the material." In Arts. 84 and 85 the author presented a closely similar view in dealing with the bending of rectangular beams; stating that the stress may approach a limiting condition in which there is a uniform longitudinal tension over one half the cross section and an equal compression over the other half. It would be desirable that the conditions should be clearly stated under which material may be expected to behave in the way assumed in these articles, and that there should be references to experiments on the subject. The point seems of importance in connection with theories of rupture. In Arts. 134 and 135 dealing with spiral springs a reference to some more complete treatment of the subject would be particularly desirable. When the author (p. 197) talks of resolving "angles about the axes ox and oy to find the horizontal and vertical components of the angular displacement * * *" his meaning may be plain to the practical man, but it certainly is not so to one habituated to a different usage of mathematical terms. Art. 138 dealing with the whirling of shafts is more satisfactory, but it might be desirable to tell the student that the formala $d^2M/dx^2 = 4\pi^2n^2wu/g$, which is really assumed in the proof, is obtainable by ordinary statics. It would also be well to refer to the possibility that the material may be strained beyond the elastic limit before "whirling" can come in (*Proceedings of the Cambridge Philosophical Society*, vol. 7, p. 283).

Chapter XI deals with several interesting problems, but the author is handicapped, as in Chapter X, by his avoid-

ance of the less elementary methods. The assumption that "hoop stresses" are the same at every point of the thickness of a thin shell—whether spherical, or cylindrical (of circular or elliptical section)—is a short way of reaching results, but it has the serious disadvantage of possessing limitations on which it throws no light. In the "oval section," Article 142, the hypothesis seems of doubtful validity under any conditions (cf. Todhunter and Pearson's History, Vol. 2, Part 1, Arts. 537–538). Articles 143–4 treating of the thick circular cylinder are less elementary and more satisfactory; but the author has to introduce the supplementary assumption that the longitudinal strain is uniform, and he commits himself to the greatest stress theory of rupture. As this is a specially good example of the difficulty of getting along without some general theory of rupture, and also of the expediency of knowing what the true theory is, the reviewer ventures on a slight addition to the text. The displacements u along the radius from the cylindrical axis, and w parallel to its length taken as the z-axis, are of the type (*Proceedings of the Cambridge Philosophical Society*, Vol. 7, p. 204)

$$u = ar + D/r, \qquad w = Az,$$

where A, a, D are constants determined from the surface conditions. Suppose these conditions to be uniform pressures p_i and p_e over the inner and outer cylindrical surfaces, whose radii are r_1 and r_2, along with uniform tension P parallel to the cylindrical axis (we may suppose P to represent the action of closed ends when such exists). Then denoting Young's modulus by E and Poisson's ratio by η, we easily find

$$u = r\left\{\frac{1-\eta}{E}\frac{r_1^2p_i - r_2^2p_e}{r_2^2 - r_1^2} - \eta\frac{P}{E}\right\} + \frac{1+\eta}{E}\frac{r_1^2r_2^2}{r}\frac{p_i - p_e}{r_2^2 - r_1^2};$$

$$w = z\left\{\frac{P}{E} - \frac{2\eta}{E}\frac{r_1^2p_i - r_2^2p_e}{r_2^2 - r_1^2}\right\}.$$

Art. 143 treats the case when P and p_e vanish, for which

$$u = \frac{p_i}{E}\frac{r_1^2}{r_2^2 - r_1^2}\{(1-\eta)r + (1+\eta)r_2^2/r\};$$

$$w = -\frac{2\eta p_i}{E}\frac{r_1^2 z}{r_2^2 - r_1^2}.$$

The corresponding principal stresses, in Todhunter and Pearson's notation (the radial, the "hoop," and the longitudinal) are

$$\widehat{rr} = -p_i(r_1/r)^2(r_2^2 - r^2)/(r_2^2 - r_1^2);$$

$$\widehat{\varphi\varphi} = p_i r_1^2(1 + r_2^2/r^2)/(r_2^2 - r_1^2); \quad \widehat{zz} = 0.$$

Everywhere $\widehat{\varphi\varphi}$ is the greatest stress, $\widehat{\varphi\varphi} - \widehat{rr}$ the maximum stress-difference, and u/r the greatest strain; and all three quantities have their greatest values over the inner cylindrical surface, where

$$\widehat{\varphi\varphi} = p_i(r_1^2 + r_2^2)/(r_2^2 - r_1^2); \quad \widehat{\varphi\varphi} - \widehat{rr} = 2p_i r_2^2/(r_2^2 - r_1^2);$$

$$u/r = (p_i/E)\{r_2^2 + r_1^2 + \eta(r_2^2 - r_1^2)\}/(r_2^2 - r_1^2).$$

Now suppose, as in Art. 143, that f is the largest safe tension in a long bar of the material; then f is a superior limit: on the greatest stress theory to $\widehat{\varphi\varphi}$, on the maximum stress difference theory to $\widehat{\varphi\varphi} - \widehat{rr}$, and on the greatest strain theory to $E(u/r)$. Thus if the limiting values allowed by the three theories to the internal pressure be respectively p_i', p_i'' and p_i''', we have

$$p_i' = f(r_2^2 - r_1^2)/(r_2^2 + r_1^2); \quad p_i'' = f(r_2^2 - r_1^2)/(2r_2^2);$$

$$p_i''' = f(r_2^2 - r_1^2)/\{r_2^2 + r_1^2 + \eta(r_2^2 - r_1^2)\};$$

and so

$$p_i'' = p_i'(r_2^2 + r_1^2)/(2r_2^2);$$

$$p_i''' = p_i' \div \{1 + \eta(r_2^2 - r_1)/(r_2^2 + r_1^2)\}.$$

As $r_2^2 > r_1^2$, clearly p_i' is always larger than p_i'' or p_i'''; or the greatest stress theory—of which alone our author takes cognizance—allows a greater internal pressure than either of the other two theories would consider safe. In a very thin shell p_i', p_i'', and p_i''' approach equality; but in a very thick shell we have ultimately $p_i'' = p_i'/2$, $p_i''' = p_i'/(1 + \eta)$.

"The use of initial internal stress in strengthening a thick tube" (Art. 145) represents a case where a good deal may depend on the theory of rupture adopted. Art. 146 assumes the stress in a revolving ring to be uniform over the cross section. Presumably this is approximately true when the diameter of the cross section is small compared to the aper-

ture of the ring, but not otherwise, and the conclusions arrived at should be correspondingly restricted.

Art. 147 dealing with the "stress in a revolving disc" is perhaps the nearest approach in the book to the use of the strict mathematical theory. In drawing conclusions as to the fastest safe speed of rotation the author again confines himself to the greatest stress theory. It happens, however, that all three rupture theories mentioned above allow the same limiting velocity in the case of a disc with a hole. In the disc without a hole the greatest stress and maximum stress difference theories likewise agree, but the greatest strain theory allows a velocity greater than that allowed by the other two theories in the ratio 1: $\sqrt{1-\eta}$. The problem of the rotating disc has had a somewhat curious history, an account of which—with a more complete treatment of the problem—will be found in the *Proceedings of the Cambridge Philosophical Society*, Vol. 7, p. 201.

Units.—Those who regard the retention of British units as a sign post on the road to destruction will regard the author as situated far on the downward track. There seems no reference to the C. G. S. system except in footnotes on pp. 3 and 53, which give the relation between tons or lbs. per square inch and kilogrammes per sq. cm. The tables of strength and of elastic moduli at the end use "tons per sq. inch," whilst the table of the "approximate weights of materials" uses "lbs. per cubic foot." In Art. 130 work is measured in "inch-pounds" per minute. In the text the author speaks of tons or lbs. per sq. inch (not ton *weights* etc.), and his attitude towards "g" when he has to use it is apt to be reserved (see Art. 72). Our own view is that so long as it is sufficiently clear what is meant, the terminology employed in referring to the unit of stress is of secondary importance. So long as "factors of safety" are as elastic (in the usual sense) as at present, it does not much matter whether the standard ton resides in London or New York.

This review has been written under the belief that what the readers of the BULLETIN are accustomed to is a critical examination of books, and not the kind of general appreciation that is so easy to write and so pleasant (for the author's uncritical friends) to read. Lest, however, the BULLETIN may occasionally fall into the hands of uncritical persons, it may be well to state explicitly that in many respects Professor Ewing's book is an excellent one, especially as an introduction to the subject. In general, it is clearly written and well arranged, and bears evidence of the author's ex-

perience as a teacher. My criticisms largely arise from the fact that the author seldom indicates the limitations and uncertainties in the mathematical methods adopted, or supplies references whence his readers might obtain the knowledge which the book does not itself supply. Even in the parts of the work dealing with experimental results one is rather struck by the paucity of references to recent work done outside the author's laboratory. The insertion of long lists of authorities in an elementary book may savor of ostentation, but the author goes too far in the opposite direction.

The book seems carefully printed. Of the few errata I have noticed the following are the chief not already referred to :—p. 84, l. 19, for "then" read "there"; p. 101, last line, for "negative" read "positive"; p. 108, lines 22 and 23, interchange "upper" and "lower" (?); p. 111. l. 6 from foot, insert "greatest" before "stress"; p. 117, case 4, the force F_x *increases* uniformly (*algebraically*) from one end to the other; p. 126, the conclusion that shearing stress is greatest at the neutral axis would follow from proof given only if breadth ζ constant; p. 135, in lines 9 and 10 of Art. 102, interchange "above" and "below"; p. 158, l. 2 below fig. 101, for 188 read 116; p. 185, l. 3 from foot, for "M" read "M_1"; p. 207, "t" is omitted in denominators of formulæ for f and f' in Art. 142; p. 208, "θ" should be shown in Fig. 136; p. 227, l. 10, for dM/da read dM_x/da.

CHARLES CHREE.

RICHMOND, SURREY,
July 2, 1900.

SCHEFFERS' DIFFERENTIAL GEOMETRY.

Anwendung der Differential- und Integralrechnung auf Geometrie. By DR. GEORG SCHEFFERS, Professor in the Darmstadt Polytechnic School. Erster Band: *Einführung in die Theorie der Curven in der Ebene und im Raum.* Leipzig, Veit and Co., 1901.

THE author of this work has already proved his capacity for writing text-books in a clear and readable manner, in the three volumes of Lie's works which he edited. The present volume is arranged and written in the same attractive and on the whole satisfactory style—for which the

author is to be particularly congratulated, inasmuch as the work is intended for students who wish to obtain a bird's-eye view of the vast subject of the applications of the infinitesimal calculus to geometry—a subject which is touched upon, more or less, by nearly every writer of mathematical text-books or papers, and which for that reason is almost inaccessible as a whole to the average student.

There is a remarkable dearth of books on this subject in the English language—practically the only sources of information for the beginner being Salmon's classical (but antiquated) works, and the various larger text-books on the calculus. Important portions of the subject are treated by all the French authors who edit a Cours d'analyse—Houël, Jordan, Picard, Laurent, etc.; but these works fail to be adapted to the wants of the beginner either from being too much condensed—as those of Jordan and Picard—or from being too diffuse, as that of Laurent. In German, the elementary text-books are Hoppe and Joachimsthal—neither of which appeals to the beginner on account of peculiarities of arrangement; while Stahl-Kommerell is much too brief and Bianchi too difficult for one who has only had a course in elementary calculus.

A great mathematical discovery is nearly always the outgrowth of a need, felt by mathematicians in general rather than by a single investigator, for the advance represented by the discovery; so that a work which coördinates and presents in a readable form the principal branches of a many-sided subject paves the way, at least, for a notable advance in the science. In this sense, Professor Scheffers deserves the thanks, both of the students of mathematics, for whom the work is especially intended, and of mathematicians in general.

The volume which has just appeared, and which we shall examine briefly, contains the first grand division of the author's subject: the theory of curves—plane curves, and space curves with their accompanying developables. The second division, which will appear within a year, will contain the general theory of surfaces.

The subjects considered in Part I. of the present volume are largely those given in any complete work on the calculus—the theory of contact of plane curves, curvature, evolute and involute, etc. It is perhaps regrettable that practically nothing is done towards the discussion of asymptotes (as is done, for example, in Houël's Calcul infinitésimal, volume 2), and the treatment of singular points is very meager.

Some sections of this part, however, treat of subjects not usually given in the current text-books, and they deserve special notice. In § 2 the writer explains what is meant by a *movement* in the plane, by showing that it is immaterial whether the operations represented by the familiar equations

$$\begin{aligned} x' &= x\cos\alpha - y\sin\alpha + a, \\ y' &= x\sin\alpha + y\cos\alpha + b, \end{aligned} \tag{1}$$

be interpreted, 1°, as a change in the position of the coordinate axes with reference to the geometrical figure under consideration, or 2°, as a change of the position of the figure with reference to the (unchanged) axes. This introduces the reader at once, and in the most natural manner, to the conception of a *transformation* of the points of the plane. In § 8 certain *differentiàl invariants* of a plane curve are defined and deduced by means of the definition of a movement contained in § 2. Any function of

$$x, \quad y, \quad \frac{dy}{dx}, \quad \frac{d^2y}{d^2x}, \quad \cdots$$

which remains unchanged, when the curve $y = f(x)$ is subjected to the movements (1), is called a differential invariant of the curve with regard to those movements.* All possible differential invariants (in this sense) are shown—in an entirely elementary manner and without reference to the theory of transformation groups—to be functions of the magnitudes

$$r, \quad \frac{dr}{d\tau}, \quad \frac{d^2r}{d^2\tau}, \cdots, \tag{2}$$

where r and τ are radius of curvature and angle of contingence, respectively.

The results of this section are applied in § 9 to establish the necessary and sufficient conditions that two plane curves shall be congruent. For since the differential invariants (2) are unchanged by a movement (1), it is clear that two curves can be congruent only when their differential invariants have the *same* values at corresponding points. It is

* The conception "differential invariant" is contained implicitly in all the older works on differential equations. Cayley, in his Theory of Invariants, and Halphen, in his "Thèse sur les invariants différentiels" (1878), established certain classes of differential invariants; but the *general* theory of differential invariants is due to Lie (*Verhand. der Gesell. der Wiss. zu Christiania*, Febr., 1875, also 1882-3; *Archiv for Math.*, 1882-3; *Math. Annalen*, vol. 24).

clear that $dr/d\tau$ is some function of τ; so that if $dr/d\tau = \omega(\tau)$ along the one curve, the necessary and sufficient condition for congruence may be seen to be that $dr/d\tau$ must be the *same* function of r along the other curve.*

All properties of a given curve therefore which are independent of the position of the coördinate axes, find their complete expression in the equation $dr/d\tau = \omega(\tau)$; therefore nothing is more natural than to introduce $dr/d\tau = \omega(\tau)$ as the intrinsic, or natural, equation to a curve.†

In § 11 it is shown that the differential invariants (2) represent, geometrically, the radii of curvature of the evolute, or of the evolute of the evolute, ···, of the given curve, as readily appears.

The §§ 14–20 of Part I. are taken up with a brief outline of the theory of ordinary differential equations in the plane: and the introduction of the conceptions *curvilinear coördinates, curve-nets*, etc. It is very fortunate for the beginner that the author treats the subject of parametric (curvilinear) coördinates in the plane very fully and clearly—carefully deducing the conditions that two curve families shall form an orthogonal system, an isometric system, etc., in the plane—as most of these results can be extended at once to curve families on a surface. It is just these sections of Part I. which will be most useful and most interesting to the beginner—as those subjects are not systematically treated in the current text-books, at least until the difficult general theory of surfaces is studied. Of especial interest are the paragraphs giving Lie's geometrical interpretation of an integrating factor, the finding of all transformations which leave areas invariant, the finding of all *conform* transformations, and the discussion of Lie's method of integrating *two* ordinary differential equations when a certain relation between their integrating factors is known.

Part II. of this volume scarcely demands a detailed discussion here, as it consists mainly of an excellent exposition of the classical theory of the space-curve, introducing the reader to the conceptions *osculating plane, contact, curvature, osculating sphere*, etc., together with the natural extension to space of the developments in Part I. concerning movements, differential invariants, etc.

In § 3, for instance, the writer explains how the (space) equations analogous to (1) define a movement, and proves incidentally Mozzi's theorem that (in general) any movement in space can be replaced by a screw movement.

* This criterion does not hold, for obvious reasons, for circles and minimal straight lines.

† That this is equivalent to Whewell's form $s = f(\tau)$ is readily verified.

Then in § 12 all differential invariants of a space curve—with regard to the movements—are determined; and they are shown to be functions of the radius of curvature r, the radius of torsion ρ, and the derivatives of these functions with respect to the arc length s. Thence are obtained the intrinsic (or natural) equations of space curves as the two independent equations which contain the differential invariants of lowest orders of the curve. In the next section the author gives Lie's method of reducing to the integration of a Riccati equation the problem of finding the finite equations of a space curve from its given intrinsic equations.

These sections, as well as § 18, 19 in which the curve is to be determined from the spherical indicatrix of its tangents, principal normals, or binormals, will offer considerable, but by no means insuperable, difficulties to the beginner.

In order to complete the development of the theory of the space curve—in particular, the theory of the evolutes or involutes of a given space curve—it is necessary to examine the curve in connection with its accompanying developable surfaces. The more important of these developables are, respectively, the envelopes of the osculating planes and of the normal planes of the curve. The author gives, therefore, in the first section of Part III. a brief discussion of the ruled surface, and, in particular, of the developable, deducing afterwards the theory of the evolutes and involutes of the space curve in his usual clear manner.

All theorems which are proved with the arc length s chosen as the variable parameter of which x, y, and z are functions, fail of course for the special curves for which s is everywhere zero. Hence the minimal curves (or minimal straight lines) demand a special discussion, and the development of the theory of these curves according to Legendre, Enneper, Weierstrass, and Lie closes the volume.

The type work of the book is excellent; very few misprints have been observed. It is furnished with an appendix containing several tables of useful formulæ, and with a good index. The author has enriched the general theory of curves by several original developments—notably in the discussion of the trajectories of a curve family * in the plane, and in the integration of the intrinsic equations † of a space curve. The historical references, given whenever a new definition is introduced, will be of great value to the

* G. Scheffers, *Ber. der math.-phys. Klasse der kgl. Sächs. Gesell. der Wiss. zu Leipzig*, October 24, 1898.

† Same publication, January 8, 1900.

reader who wishes to penetrate deeper into this branch of the science. The figures in the text are drawn with great care, and the illustrative examples worked out fully and clearly. The reader of English text-books will miss the long lists of problems usually given at the ends of the chapters in such works; but we predict for the book a very useful career in the lecture room, especially in the hands of an energetic teacher who can supply himself with an abundance of illustrative problems.

J. M. PAGE.

UNIVERSITY OF VIRGINIA,
November 11, 1900.

NOTES.

A NEW edition of the Annual Register of the Society will be issued in January. Forms for furnishing necessary information have been sent to each member, and a prompt response will be of great assistance to the Secretary.

AT the annual general meeting of the London mathematical society held November 8, 1900, the following officers were elected: Dr. E. W. HOBSON, president; Lord KELVIN, Professor W. BURNSIDE, and Major P. A. MACMAHON, vice-presidents; Dr. J. LARMOR, treasurer; Mr. R. TUCKER, and Professor A. E. H. LOVE, honorary secretaries; Mr. J. E. CAMPBELL, Lieut.-Col. A. C. CUNNINGHAM, Professor E. B. ELLIOTT, Dr. J. W. L. GLAISHER, Professor M. J. M. HILL, Messrs. A. B. KEMPE, H. M. MACDONALD, A. E. WESTERN and E. T. WHITTAKER, additional members of the council. The subject of LORD KELVIN'S address as retiring president of the society was "The transmission of force through a solid."

AT the anniversary meeting of the Royal Society of London, on November 30, Sir WILLIAM HUGGINS was elected president. Among the new members of the council is Professor E. B. ELLIOTT. A Royal medal was presented to Major P. A. MACMAHON for his contribution to mathematical science.

THE National academy of sciences held its autumn meeting at Brown University, Providence, R. I., on November

13. The following mathematical and physical papers were among those read :

J. TROWBRIDGE : "Investigations of light and electricity with the aid of a battery of twenty thousand cells."

S. L. PENFIELD : "Stereographic projection and some of its possibilities from a graphical standpoint."

T. C. MENDENHALL : "Note on the energy of recent earthquakes."

H. A. ROWLAND : "On the explanation of inertia and gravitation by means of electrical phenomena."

C. BARUS : "On stability of vibration and on vanishing resonance."

THE Royal Academy of Belgium proposes the following prize questions for the year 1901 :

1. Find the form of the principal terms introduced by the elasticity of the earth's shell into the formulas for the change of obliquity and of longitude. Value of the prize, 800 francs.

2. An important contribution is to be made to the theory of mixed forms in any number of series of variables, and the results are to be applied to the geometry of the corresponding space. Value of the prize, 600 francs.

The memoirs may be written in French or Flemish, and should be sent before August 1, 1901, under the usual conditions as to anonymity to the permanent secretary of the academy in Brussels.

UNIVERSITY OF PARIS. The following courses in mathematics are among the announcements of the faculty of sciences for the academic year 1900–1901 :—First semester : —By Professor G. DARBOUX : Fundamental principles of infinitesimal geometry especially with reference to the theory of triple systems of orthogonal surfaces, two hours. —By Professor E. GOURSAT : The operations of the differential and integral calculus, and analytic functions, two hours.—By Professor P. APPELL : The general laws of equilibrium and motion, two hours.—By Professor H. POINCARÉ : The motions of celestial bodies about their centers of gravity, two hours.—By Professor BOUSSINESQ : On the internal friction of fluids, two hours.—By Professor G. KŒNIGS : Kinematics of solid and deformable bodies, with application to the study of machines, two hours.—By Professor L. RAFFY : Elements of analysis and mechanics, two hours.—By Dr. M. ANDOYER : General theory of planetary perturbations, one hour.—Conferences by Pro-

fessors RAFFY, HADAMARD, and PUISEUX, and Messrs. ANDOYER and BLUTEL.—The preliminary announcements for the second semester include courses by Professor E. PICARD on algebraic functions, by Professor E. GOURSAT on differential equations, by Professor P. APPELL on analytical mechanics, by Professor G. KŒNIGS on the study of machines, and by Professor L. RAFFY on the differential equations of mechanics and physics.

OXFORD UNIVERSITY. Advanced mathematical courses for the Michaelmas term of the current academic year are announced as follows:—By Professor E. B. ELLIOTT: Theory of numbers, two hours; substitutions and resolvents, one hour.—By Professor W. ESSON: Analytic geometry of plane curves, two hours; synthetic geometry of plane curves, one hour.—By Professor A. E. H. LOVE: Gravitational attraction and theory of the potential, three hours; theory of sound, one hour.—By Mr. J. E. CAMPBELL: Differential equations, two hours.—By Mr. H. T. GERRANS: Three-dimensional rigid dynamics, two hours.

CAMBRIDGE UNIVERSITY. Mr. A. N. WHITEHEAD and Mr. A. BERRY are the examiners, and Mr. J. G. LEATHAM and Mr. J. H. GRACE the moderators for Part I. of the mathematical tripos, 1901. Professor H. LAMB, Mr. H. W. RICHMOND, Mr. H. F. BAKER and Mr. H. M. MACDONALD are the examiners for Part II. of the same tripos.

THE Smith's prizes for 1900 have been granted to J. F. CAMERON for his essay: "The molecules as electric oscillators," and to R. W. H. T. HUDSON for "Differential equations of the second order and their singular solutions." Mr. Hudson was senior wrangler and Mr. Cameron second wrangler in 1898.

THE library of the late DR. M. C. VERLOREN VAN THEMAAT, containing a number of rare mathematical works, will be sold at auction by Frederick Muller and Company, Doelenstraat 10, Amsterdam, Holland, on December 18–19.

IN a pamphlet of forty-seven pages, Ueber die Nomographie von M. d'Ocagne, Leipzig, Teubner, 1900, Professor F. SCHILLING presents the salient ideas of M. d'Ocagne's book (reviewed in the BULLETIN, volume 5, p. 362) under the conviction of its great utility for all branches of applied mathematics. As this utility seems generally ad-

mitted, we may hope that a series of abaci for practical use will soon be accessible.

ACCORDING to *Nature*, Dr. CULLIS, professor of mathematics, at Hartley College, Southampton, has been appointed professor of mathematics at the Presidency College, Calcutta; and Mr. J. F. HUDSON, late lecturer in mathematics at Jesus College, Oxford, has been appointed professor of mathematics at Hartley College, Southampton.

PROFESSOR FRANK MORLEY of Johns Hopkins University has recently been appointed editor of the *American Journal of Mathematics*.

PROFESSOR A. TRESSE has been appointed professor of mathematics at the collège Rollin, Paris.

DR. H. LORENZ, of Halle, has been made director of the physical and technical institute of the university of Göttingen.

PROFESSOR G. OLTRAMARE, the venerable dean of the faculty of sciences of the university of Geneva, has retired from the chair of higher mathematics at the age of eighty-four, after a tenure of fifty-two years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BONOLA (R.). Sulla teoria delle parallele e sulla geometria non-Euclidea. Bologna, 1900. 8vo. 80 pp., 2 plates.

BURONI (S.). See LAVAGGI (L.).

CANTOR (M.). Vorlesungen über Geschichte der Mathematik. (In 3 Bänden.) Vol. III: Von 1668–1758. Abteilung 1: Von 1668–1699. 2te Auflage. Leipzig, Teubner, 1900. 8vo. 261 pp. M. 6.60

ERMAKOF (V. P.). Lectures on the integral calculus. (Russian.) Parts 1 and 2. Kief, 1900. 8vo. 350 pp. R. 3.00

FRENET (F.). Collection of problems in the infinitesimal calculus. Part 2: Integral calculus; part 3: Miscellaneous problems and appendix. Translated into Russian from the 5th French edition by A. P. Nenashef. (Russian.) Moscow, 1900. 8vo. 4 + 240 pp. R. 2.50

FRIEDRICH (M.). Katechismus der analytischen Geometrie. 2te Auflage, durchgesehen und verbessert von E. Riedel. Leipzig, Weber, 1900. 12mo. 8 + 217 pp. (Weber's illustrirte Katechismen, No. 116.) Cloth. M. 3.00

HAGEN (J. G.). Synopsis der höheren Mathematik. Vol. 3. Lieferung 2. Berlin, Dames, 1900. 4to. M. 5.00

HENKE (R.). See SCHLÖMILCH (O.).

JAHRBUCH über die Fortschritte der Mathematik. Begründet von C. Ohrtmann. Unter besonderer Mitwirkung von F. Müller, A. Wangerin und G. Wallenberg herausgegeben von E. Lampe. Vol. 29: Jahrgang 1898. (In 3 Heften.) Heft 2. Berlin, Reimer, 1900. 8vo. Pp. 417–576. M. 5.00

LAMPE (E.). See JAHRBUCH.

LAVAGGI (L.). Calcolo infinitesimale. Lezioni [dettate nell' anno] 1899–1900 nella r. università di Parma, compilate per cura di S. Buroni. Disp. 71–102 (ultima). Parma, Zafferri, 190. 8vo. Pp. 561–815.

LÜBECK (O.). Algebraische Analysis. Unterweisungen und Beispiele. Strelitz, Hittenkofer, 1900. 8vo. 30 pp. (Unterrichtswerke für Selbstunterricht und Bureaugebrauch, Methode Hittenkofer.) M. 2.00

MÜLLER (F.). Mathematisches Vokabularium französisch-deutsch und deutsch-französisch, enthaltend die Kunstausdrücke aus der reinen und angewandten Mathematik. 1ste Hälfte. Leipzig, Teubner, 1900. 8vo. 9 + 132 pp. M. 8.00

——. See JAHRBUCH.

NENASHEF (A. P.). See FRENET (F.).

PENSA (A.). Sull' influenza di alcune singolarità di superficie sul genere numerico e sul bigenere P_2 con applicazioni alla determinazione di superficie razionali di quinto ordine. Mondovi, Vescovile, 1900. 8vo. 28 pp.

PIZZARELLO (D.). Sulle funzioni trascendenti intere; memoria presentata come tesi di laurea nella r. università di Roma il 14 novembre, 1899. Messina, Tipografia dell' *Epoca*, 1900. 8vo. 70 pp.

RIEDEL (E.). See FRIEDRICH (M.).

SCHLÖMILCH (O.). Uebungsbuch zum Studium der höheren Analysis. Teil II: Aufgaben aus der Integralrechnung. 4te Auflage, bearbeitet von R. Henke. Leipzig, Teubner, 1900. 8vo. 8 + 448 pp. M. 9.00

WALLENBERG (G.), WANGERIN (A.). See JAHRBUCH.

WILCZYNSKI (E. J.). On continuous binary linearoid groups, and the corresponding differential equations and Λ functions. (*American Journal of Mathematics*, Vol. 22, pp. 191–225.) 4to.

——. An application of group theory to hydrodynamics. (*Transactions of the American Mathematical Society*, Vol. 1, pp. 339–352.) 4to.

——. Poetry and mathematics. Berkeley, University Press, 1900. 8vo. 16 pp.

II. ELEMENTARY MATHEMATICS.

BARONI (E.). Sui metodi elementari per la risoluzione dei problemi geometrici. Bologna, Zanichelli, 1900. 8vo. 23 pp.

BERRY (T. W.) and VINE (G. F.). How to work deductions in Euclid. London, Simpkin, 1900. 12mo. 112 pp. 2 s.

DUPUIS (N. F.). Principles of elementary algebra. Revised and corrected edition. New York, Macmillan, 1900. 12mo. 7 + 336 pp. Cloth. $1.10

DURFEE (W. P.). Elements of plane trigonometry. Boston, Ginn, 1900. 16mo. 6 + 105 pp. Cloth. $0.80

FAUDELLA e PANTALEO. Aritmetica e geometria per la quarta classe elementare. Torino, Paravia, 1900. 16mo. 128 pp. Fr. 0.60

——. Aritmetica e geometria per la quinta classe elementare. Torino, Paravia, 1900. 16mo. 99 pp. Fr. 0.60

GAZZANIGA (E. P.). Aritmetica generale: divisioni ordinarie, speciali e mobili, configurazioni numeriche e numeri primi, divisioni e frazioni periodiche, nuovi studî e teoremi. Bergamo, Bolis, 1900. 8vo. 4 + 213 pp. Fr. 5.00

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HOLZMANN (A.) und MASSINGER (R.). Geometrische Anschauungslehre (in 3 Teilen) im Anschluss an den Lehrplan der badischen Realschulanstalten. Teil I: Erster Abschnitt der ebenen Gebilde; Pensum der Klasse IV; 32 pp. Teil II: Zweiter Abschnitt der ebenen Gebilde; Pensum der Klasse V; 30 pp. 2te Auflage. Karlsruhe, Reiff, 1900. 8vo. Boards. M. 0.60

JACQUET (E.) et LACLEF (A.). Cours d'arithmétique théorique et pratique, avec de nombreux exercices, problèmes, etc., à l'usage des écoles normales d'instituteurs et d'institutrices, des écoles primaires supérieures, des écoles professionnelles et des candidats au brevet supérieur. Paris, Nathan, 1901. 18mo. 8 + 317 pp.

JULING (G.). Fünfstellige Logarithmentafeln für Schüler. Leipzig, Berger, 1900. 8vo. 144 pp. Cloth. M. 1.20

KUTNEWSKY (M.). See MÜLLER (H.).

LACLEF (A.). See JACQUET (E.).

LICHTBLAU (W.) und WIESE (B.). Sammlung geometrischer Rechenaufgaben, zum Gebrauch an Seminarien, sowie zum Selbstunterricht bestimmt und herausgegeben. 2te Auflage. Breslau, Hirt, 1900. 8vo. 164 pp. M. 1.75

LIEBER (H.) und LÜHMANN (F. VON). Leitfaden der Elementar-Mathematik. Neu herausgegeben von C. Müsebeck. Teil I: Ausgabe B für Realgymnasien, Oberreal- und Realschulen. Planimetrie. Einführung in die Trigonometrie und Stereometrie. (Lehraufgabe der Quarta bis Untersekunda.) Berlin, Simion, 1900. 8vo. 5 + 109 pp. M. 1.60

LÜHMANN (F. VON). See LIEBER (H.).

MASSINGER (R.). See HOLZMANN (A.).

MÜLLER (H.) und KUTNEWSKY (M.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. Ausgabe A, für Gymnasien, Realgymnasien und Oberrealschulen. Teil I. Leipzig, Teubner, 1900. 8vo. 8 + 315 pp. Cloth. M. 2.80

——. Ausgabe B, für Realschulen. Teil I. Leipzig, Teubner, 1900. 8vo. 8 + 289 pp. M. 2.60

MÜSEBECK (C.). See LIEBER (H.).

NUÑEZ (M. G.). Tratado completo de matematicas. Cuaderno 1: Operaciones fundamentales con numeros enteros. Madrid, 1900. 4to. Pp. 1–82. Fr. 3.50

PANTALEO. See FAUDELLA.

PAYNE (E. L.). Principles and methods of college algebra. Emporia, Kan., M'Cord & M'Cord, 1900. 12mo. 56 pp. $0.25

RIEDEL (E.). Katechismus der Planimetrie, mit einem Anhang über harmonische Teilung, Potenzlinien und das Berührungsproblem des Apollonius. Leipzig, Weber, 1900. 12mo. 10 + 346 pp. (Weber's illustrirte Katechismen, No. 225.) Cloth. M. 4.00

SERRET (J. A.). Traité de trigonométrie. 8e édition. Paris, Gauthier-Villars, 1900. 8vo. 10 + 336 pp. Fr. 4.00

VINE (G. F.). See BERRY (T. W.).

WIESE (B.). See LICHTBLAU (W.).

III. APPLIED MATHEMATICS.

ADAMCZIK (J.). Compendium der Geodäsie. Wien, Deuticke, 1900. 8vo. 8 + 515 pp. M. 10.00

BURONI (S.). See CARDANI (P.).

CALDARERA (F.). Corso di meccanica razionale. Vol. I: Cinematica. Palermo, 1900. 8vo. 329 pp. Fr. 12.50

CARDANI (P.). Fisica matematica. Lezioni [dettate nell'anno scolastico] 1899–1900 nella r. università di Parma e compilate per cura di S. Buroni. Disp. 31–40 (ultima). Parma, Zafferri, 1900. 8vo. Pp. 241–320.

CASTLE (F.). Workshop mathematics. Part I. London and New York, Macmillan, 1900. 12mo. 8 + 154 pp. Cloth. $0.35

——. Part II. London and New York, Macmillan, 1900. 12mo. 9 + 177 pp. Cloth. $0.35

DÉCOMBE (L.). La célérité des ébranlements de l'éther. Evreux, Hérissey, 1900. 16mo. 96 pp. (Scientia, physique mathématique, No. 9.)

EMTAGE (W. T. A.). Elementary mechanics of solids. London and New York, Macmillan, 1900. 12mo. 338 pp. $0.60

FÖPPL (A.). Vorlesungen über die technische Mechanik. Vol. 2: Graphische Statik. Leipzig, Teubner, 1900. 8vo. 10 + 452 pp. Cloth. M. 10.00

FUHRMANN (A.). Anwendungen der Infinitesimalrechnung in den Naturwissenschaften, im Hochbau und in der Technik. Lehrbuch und Aufgabensammlung. (In 6 Teilen.) Teil I: Naturwissenschaftliche Anwendungen der Differentialrechnung. 2te Auflage. Berlin, Ernst, 1900. 8vo. 18 + 239 pp. M. 6.00

GUILHAUMON (J. B.). Résumé de théorie du navire. 2e édition. Paris, Berger-Levrault, 1900. 8vo. 84 pp., 2 plates. Fr. 2.50

HERMAN (R. A.). A treatise on geometrical optics. London, Clay, 1900. 8vo. 354 pp. 10s.

HERTZ (H.). Electric waves; researches on the propagation of electric action with finite velocity through space. Authorized English translation by D. E. Jones, with a preface by Lord Kelvin. London, Macmillan, 1900. 8vo. 298 pp. 10s.

KELVIN. See HERTZ (H.).

JONES (D. E.). See HERTZ (H.).

LAUENSTEIN (R.). Die Festigkeitslehre. Elementares Lehrbuch für den Schul- und Selbstunterricht sowie zum Gebrauch in der Praxis, nebst einem Anhang enthaltend Tabellen der Potenzen, Wurzeln, Kreisumfänge und Kreisinhalte. 6te Auflage. Stuttgart, Bergsträsser, 1900. 8vo. 6 + 179 pp. M. 4.40

LONEY (S. L.). Elements of hydrostatics. London, Clay, 1900. 12mo. 270 pp. (Pitt press mathematical series.). 4s. 6d.

ROSENBERG (F.). First stage mechanics of solids. For the elementary examination of the Science and art department. 3d edition. London, Clive, 1900. 12mo. 320 pp. (Organized science series.) 2s.

SLATE (F.). The principles of mechanics. An elementary exposition for students of physics. Part 1. New York, The Macmillan Co., 1900. 12mo. 10 + 299 pp. Cloth. $1.90

STREISSLER (J.). Elemente der darstellenden Geometrie für Oberrealschulen. 4te Auflage. Brünn, Winiker, 1900. 8vo. 4 + 162 pp., 4 plates. M. 2.60

VITI (R.). Sulla teoria matematica della previdenza; seconda conferenza tenuta il 17 marzo 1900 alla r. accademia dei ragionieri di Bologna. Roma, Giovanni, 1900. 8vo. 15 pp.

WILLIAMSON (B.) and TARLETON (F. A.). Elementary treatise on dynamics containing applications to thermodynamics, with numerous examples. 3d revised and enlarged edition. London and New York, Longmans, Green & Co., 1900. 12mo. 16 + 559 pp. Cloth. $3.50

ZEUNER (G.). Technische Thermodynamik. 2te Auflage; zugleich 4te Auflage der "Grundzüge der mechanischen Wärmetheorie." Vol. I: Fundamentalsätze der Thermodynamik; Lehre von den Gasen. Leipzig, Felix, 1900. 8vo. 16 + 436 pp. M. 13.00